Announcements

1) We're covering sections 6.3 – 6.6 in text.

Definitions:

1) $U \in M_n(\mathbb{C})$ is said to be unitary if $\boxed{U^*U = UU^* = I_n}$

2) $A \in M_n(\mathbb{C})$ is said to be self-adjoint if $\boxed{A = A^*}$

Example 1:

$$U = \begin{pmatrix} 1/\sqrt{2} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\left(\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right)$$

$$= -\frac{i}{2} + \frac{i}{2} = 0$$

$\Rightarrow$ U is unitary

$$A = \begin{pmatrix} 2 & \pi + i \\ \pi - i & -6 \end{pmatrix}$$

is self-adjoint

(the diagonal must be real)

Proposition: (eigenvalues)

a) If $U \in M_n(\mathbb{C})$ is $\boxed{\text{unitary}}$ and λ is an eigenvalue of U, then

$$\boxed{|\lambda| = 1}$$

b) If $A \in M_n(\mathbb{C})$ is $\boxed{\text{self-adjoint}}$ and λ is an eigenvalue, then

$$\boxed{\lambda \in \mathbb{R}}$$

proof: a) Let λ be an eigenvalue for U and let $x \in \mathbb{C}^n$ be an eigenvector.

$$\|x\|_2^2 = \langle x, x \rangle$$

$$= \langle I_n x, x \rangle$$

$$= \langle U^* U x, x \rangle \quad \text{(U unitary)}$$

$$= \langle Ux, Ux \rangle$$

$$= \langle \lambda x, \lambda x \rangle = |\lambda|^2 \cdot \|x\|_2^2$$

Then since $x \neq 0$,

dividing by $\|x\|_2^2$

gives $|\lambda|^2 = 1$

$\Rightarrow |\lambda| = 1.$

b) Let γ be an eigenvalue for A. Let x be an eigenvector associated to γ.

Then

$$\gamma \|x\|_2^2$$

$$= \gamma \langle x, x \rangle$$

$$= \langle \gamma x, x \rangle$$

$$= \langle Ax, x \rangle$$

$$= \langle x, A^* x \rangle$$

$$= \langle x, Ax \rangle$$ (since A self-adjoint)

$$= \langle x, \gamma x \rangle$$

$$= \bar{\gamma} \langle x, x \rangle = \bar{\gamma} \|x\|_2^2$$

Since $x \neq 0$, dividing by $\|x\|_2^2$ gives

$$\boxed{\gamma = \overline{\gamma}}$$ and so

γ is real. $\square$

Analogy: ($\mathbb{C}$)

Normal Matrix $\sim$ Complex Number

Self-Adjoint Matrix $\sim$ Real Number

Unitary Matrix $\sim$ Complex number of modulus 1.

Orthogonal Projections

Recall that if

$\{x_1, \ldots, x_n\}$ is an

orthonormal (really only need

orthogonal) basis for

$\mathbb{C}^n$, we can define an

orthogonal projection onto

$W = \text{span}\{x_1, x_2, \ldots, x_k\}$

by, $\forall \; x \in \mathbb{C}^n$,

$$\boxed{Px = \sum_{i=1}^{k} \langle x, x_i \rangle x_i}$$

By Gram-Schmidt, every orthogonal projection onto a subspace W arises in such a manner.

If $1 \le i \le k$,

$$\underline{Px_i = x_i}.$$

If $k < i \le n$, then

$$\underline{Px_i = 0}$$

Then P has only two eigenvalues: $\lambda_0 = 0$ (multiplicity $n-k$) and $\lambda_1 = 1$ (multiplicity k).

Then since $\{x_i\}_{i=1}^{n}$ is an orthonormal basis of eigenvectors of P, P is unitarily diagonalizable:

$$\boxed{P = U D U^*}$$

where U is unitary and D is diagonal with only zeros and ones on the diagonal.

Then

$$P^* = (UDU^*)^*$$

$$= U D^* U^*$$

$$= U D U^* \quad (D \text{ real})$$

$$= P.$$

Moreover,

$$
\begin{aligned}
P^2 &= (UDU^*)(UDU^*) \\
&= UD(U^*U)DU^* \\
&= UDI_nDU^* \\
&= UD^2U^* \\
&= UDU^* \quad \text{(since all entries of } D \text{ are either zero or one)} \\
&= P
\end{aligned}
$$

This is a complete characterization of orthogonal projections:

$P \in M_n(\mathbb{C})$ is an orthogonal projection if and only if

$$\boxed{P = P^2 = P^*}$$

End of the Class:

What can we say for any matrix? Don't want to assume special properties.

We already know not every $A \in M_n(\mathbb{C})$ is diagonalizable.

Theorem: Let $A \in M_n(\mathbb{C})$.

Then $\exists$ unitary matrix $U \in M_n(\mathbb{C})$ and an upper-triangular matrix $T \in M_n(\mathbb{C})$ with

$$\boxed{A = UTU^*}$$

Proof: (idea)

The proof is by induction.

$n=1$ is trivial.

Assume the result is true for $k=n-1$.

Let λ be an eigenvalue for A^*; let x be an associated eigenvector.

Suppose $\|x\|_2 = 1$.

Let $y \in \{x\}^\perp$.

Claim: $Ay \in \{x\}^\perp$.

$$\begin{aligned}
&\langle x, Ay \rangle \\
&= \langle A^* x, y \rangle \\
&= \langle \lambda x, y \rangle \\
&= \lambda \langle x, y \rangle \\
&= 0 \quad \text{since } y \in \{x\}^\perp.
\end{aligned}$$

Then $A(\{x\}^\perp) \subseteq \{x\}^\perp$;

and $\{x\}^\perp$ has complex

dimension $(n-1)$.

Idea: Apply induction

to $A\big|_{\{x\}^\perp}$ and

obtain a basis $\{x_1, \dots, x_{n-1}\}$

of $\{x\}^\perp$ such that "A is

upper triangular with respect

to $\{x_1, \dots, x_{n-1}\}$".

We can choose

$\{x_1, \ldots, x_{n-1}\}$ to be

orthonormal.

Then, with respect

to the basis

$$\mathcal{B} = \{x_1, \ldots, x_{n-1}\} \cup \{x\},$$

A is upper triangular.